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Sliding Mode Control with Optimal Sliding Surfaces for Missile Autopilot Design

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A new method is introduced to design sliding mode control with optimally selected sliding surfaces for a class of nonlinear systems. The nonlinear systems are recursively approximated as linear time-varying systems, and corresponding time-varying sliding surfaces are designed for each approximated system so that a given optimization criterion is minimized. The control input, which is designed by using an approximated system, is then applied to the nonlinear system. The method is used to design an autopilot for a missile where the design requirement is to follow a given acceleration command. The sliding surface is selected such that a performance index formed as a function of angle of attack, pitch rate, and velocity error is minimized. It is shown that the response of the approximating sequence of linear time-varying systems converges to the response of the missile.

I. Introduction

OPTIMAL sliding surfaces designed to minimize a performance criterion for linear time-invariant (LTI) systems have been studied and reported on by various authors.[1–3] In these studies, a linear sliding surface, which passes through the origin, was used with the slope of this surface designed such that a given (or desired) cost function is minimized. The design begins by transforming the system into the regular form. In regular form, the system is divided into two subsystems; in one the control explicitly appears, and in the other the control terms do not appear. The stability of the uncontrolled part determines the overall stability, and the choice of sliding surface is based on this part.

Time-varying sliding surfaces have also been explored[4,5] in terms of their stability and advantages. In addition to these studies, the optimal sliding surface design for linear time-varying (LTV) systems was investigated[6] and was, in essence, a time-varying extension of sliding surface design for LTI systems.

The sliding surface design procedure for nonlinear systems, on the other hand, becomes more complicated in all aspects. For instance, the state transformation, which transforms a nonlinear system into a suitable form (such as regular form or phase canonic form), is generally nonlinear, and the required transformation may not always exist for all nonlinear systems. The existence of nonlinear state transformations to obtain regular form and phase canonic form is explored in Refs. 7 and 8. Another difficulty is that the sliding surface may not be easily selected as in the case of LTI or LTV systems because the stability of a nonlinear system is now to be guaranteed. As a result, one may not always find a sliding surface for nonlinear systems that satisfies a desired performance or even guarantees system stability.

Although different design methods, each of which uses linear or nonlinear sliding surfaces, have been suggested for nonlinear systems,[9,10] the optimal selection of the sliding surface for a nonlinear system has not been investigated yet. In this work, a new method is suggested for choosing an optimal sliding surface for a class of nonlinear systems. The nonlinear system of concern is represented by

$$\dot{x} = A(x)x + B(x)u \tag{1}$$

where $x \in R^n$, $u \in R^m$, $A(x) \in R^{n\times n}$, and $B(x) \in R^{n\times m}$. First, a successive approximation approach is presented to approximate Eq. (1) as a sequence of LTV systems, that is,

$$\dot{x}^{[i]} = A\left[x^{[i-1]}(t)\right]x^{[i]} + B\left[x^{[i-1]}(t)\right]u^{[i]} \tag{2}$$

and then the results of Refs. 6 and 11 are used to select the optimal sliding surface for the approximated system. The control input, which is generated from the approximated system, is then applied to the original nonlinear equations.

The controller design method from this approximation approach is used to design an autopilot for a missile. Readers can find the application of sliding mode control (SMC) technique to aircrafts and flight control systems in Refs. 12–14 and the references therein. In these studies, generally linearized models and, thus, linear sliding surfaces are used. In contrast to these works, the missile dynamics studied here is nonlinear, and the sliding surface is designed as LTV manifolds. Moreover, slopes of the sliding surface are selected optimally to minimize a given performance index that is constructed as a function of angle of attack, pitch rate, and velocity error. By using LTV approximate systems, the difficulty of finding nonlinear state transformations is removed.

The paper is organized as follows: The approximation theory and convergence conditions for nonlinear state feedback and SMC are derived in Sec. II. Section III briefly explains SMC theory for LTV systems and presents the optimal sliding surface design method. In Sec. IV, the theory is applied to nonlinear missile dynamic equations as studied in Ref. 15, and an acceleration autopilot is designed for the pitching motion of the missile. To present the main idea, it is assumed that there is no uncertain parameter in the missile model.

II. Linear Approximations of Nonlinear Systems

Linearization of nonlinear systems about an equilibrium (or operating) point via Taylor's series expansion method is well defined and commonly used to study the behavior of the system and to design a suitable controller that yields desired motion for the linearized model. It is clear that the behavior of the linearized model does not represent the global motion of the nonlinear system and that the controller that is designed by using the linearized model is only effective in the neighborhood of the equilibrium point. To extend the operating region of the controller, different linearized models are derived at different operating points, and the gains of controller are tuned as the system passes through the specified operating trajectory.

In this section, we shall present another approximation technique for a class of nonlinear systems represented by Eq. (1). The technique recursively constructs LTV approximations (apart from the

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first approximation) of the system. The approximating sequence of Eq. (1) is formed by

$$\dot{x}^{[i]} = A\left[x^{[i-1]}(t)\right]x^{[i]} + B\left[x^{[i-1]}(t)\right]u^{[i]}, \qquad x^{[i]}(t_0) = x_0 \tag{3}$$

where the first approximation is

$$\dot{x}^{[1]} = A(x_0)x^{[1]} + B(x_0)u^{[1]}, \qquad x^{[1]}(t_0) = x_0 \tag{4}$$

The first approximation of the nonlinear system (1) is an LTI system, and afterwards the approximating sequence results in LTV systems. Before giving the convergence conditions of Eq. (3), consider the unforced part of Eq. (1), that is,

$$\dot{x} = A(x)x \tag{5}$$

with its approximating sequence given by

$$\dot{x}^{[i]} = A\left[x^{[i-1]}(t)\right]x^{[i]}, \qquad x^{[i]}(t_0) = x_0$$
$$\dot{x}^{[1]} = A(x_0)x^{[1]}, \qquad x^{[1]}(t_0) = x_0 \tag{6}$$

Let $\Phi^{[i-1]}(t, t_0)$ be the transition matrix generated by $A[x^{[i-1]}(t)]$. It is well known (see Ref. 16) that

$$\left\|\Phi^{[i-1]}(t, t_0)\right\| \le \exp\left(\int_{t_0}^{t} \mu\left\{A\left[x^{[i-1]}(\tau)\right]\right\} \mathrm{d}\tau\right) \tag{7}$$

where $\mu[A(\cdot)]$ is the logarithmic norm of $A(\cdot)$ and is defined by

$$\mu[A(\cdot)] := \lim_{h\to 0^+} \{[\|I + hA(\cdot)\| - 1]/h\}$$

where $\|\cdot\|$ is the standard spectral (induced) matrix norm. Here $\mu[A(\cdot)]$ can also be defined as

$$\mu[A(\cdot)] := \tfrac{1}{2}\max \lambda[A(\cdot) + A(\cdot)^T]$$

The following lemma is for an estimate of $\Phi^{[i-1]}(t, t_0) - \Phi^{[i-2]}(t, t_0)$.

Lemma 1: Suppose that $\mu[A(x)] \le \kappa_1$, for all x and that

$$\|A(x) - A(y)\| \le \kappa_2\|x - y\|, \qquad \forall x, y \in R^n$$

Then

$$\left\|\Phi^{[i-1]}(t, t_0) - \Phi^{[i-2]}(t, t_0)\right\| \le \kappa_2 e^{\kappa_1(t-t_0)}(t - t_0) \times \sup_{s\in[t_0,t]} \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|$$

Proof: $\Phi^{[i-1]}(t, t_0)$ and $\Phi^{[i-2]}(t, t_0)$ are solutions of the respective equations

$$\dot{x}^{[i]} = A\left[x^{[i-1]}(t)\right]x^{[i]}, \qquad x^{[i]}(t_0) = x_0$$
$$\dot{x}^{[i-1]} = A\left[x^{[i-2]}(t)\right]x^{[i-1]}, \qquad x^{[i-1]}(t_0) = x_0$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x^{[i]}(t) - x^{[i-1]}(t)\right] = A\left[x^{[i-1]}(t)\right]\left[x^{[i]}(t) - x^{[i-1]}(t)\right] + \left\{A\left[x^{[i-1]}(t)\right] - A\left[x^{[i-2]}(t)\right]\right\}x^{[i-1]}(t)$$

and then

$$x^{[i]}(t) - x^{[i-1]}(t) = \int_{t_0}^{t} \Phi^{[i-1]}(t, s)\left\{A\left[x^{[i-1]}(s)\right] - A\left[x^{[i-2]}(s)\right]\right\}x^{[i-1]}(s)\,\mathrm{d}s$$

Since

$$x^{[i]}(t) = \Phi^{[i-1]}(t, t_0)x_0$$
$$x^{[i]}(t) - x^{[i-1]}(t) = \left[\Phi^{[i-1]}(t, t_0) - \Phi^{[i-2]}(t, t_0)\right]x_0$$

then

$$\begin{aligned}\left\|x^{[i]}(t) - x^{[i-1]}(t)\right\| &\le \int_{t_0}^{t} \exp\left(\int_{s}^{t} \mu\left\{A\left[x^{[i-1]}(\tau)\right]\right\}\mathrm{d}\tau\right)\kappa_2\left\|x^{[i]}(s) - x^{[i-1]}(s)\right\| \\ &\quad\times \exp\left(\int_{t_0}^{s} \mu\left\{A\left[x^{[i-2]}(\tau)\right]\right\}\mathrm{d}\tau\right)\|x_0\|\,\mathrm{d}s \\ &\le \kappa_2 e^{\kappa_1(t-t_0)}(t - t_0)\sup_{s\in[t_0,t]}\left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|\,\|x_0\|\end{aligned}$$
□

Now define

$$\xi^{[i]}(t) := \sup_{s\in[t_0,t]}\left\|x^{[i]}(s) - x^{[i-1]}(s)\right\| \tag{8}$$

Then,

$$\xi^{[i]}(t) \le \kappa_2 e^{\kappa_1(t-t_0)}(t - t_0)\|x_0\|\xi^{[i-1]}(t)$$

Suppose that

$$\gamma_1(t) := \kappa_2 e^{\kappa_1(t-t_0)}(t - t_0)\|x_0\| < 1 \tag{9}$$

Then

$$\xi^{[i]}(t) \le \gamma_1(t)\xi^{[i-1]}(t)$$

and so

$$\xi^{[i]}(t) \le \gamma_1^{i-2}(t)\xi^{[2]}(t) \tag{10}$$

Theorem 1: Let κ_1 and κ_2 be finite numbers and $A(x)$ satisfy

$$\mu[A(x)] \le \kappa_1, \qquad \forall x \in R^n \tag{A.1}$$

$$\|A(x) - A(y)\| \le \kappa_2\|x - y\|, \qquad \forall x, y \in R^n \tag{A.2}$$

and suppose that

$$\gamma_1(T) = \kappa_2 e^{\kappa_1(T-t_0)}(T - t_0)\|x_0\| < 1$$

Then Eq. (5) has a unique solution on $[t_0, T]$ that is given by the limit of the solutions of the approximating equations (6) on $C([t_0, T], R^n)$.

Proof: The proof follows directly from Eq. (10) because this implies that $x^{[i]}(t)$ is a Cauchy sequence in the Banach space $C([t_0, T], R^n)$. □

To see the effect of control term u on the approximating sequence, let us first consider a nonlinear state feedback control, that is,

$$u = K(x)x \tag{11}$$

which yields

$$\dot{x} = A(x)x + B(x)K(x)x := A(x)x + \hat{B}(x)x \tag{12}$$

Then the approximating sequence is

$$\dot{x}^{[i]} = A\left[x^{[i-1]}(t)\right]x^{[i]} + \hat{B}\left[x^{[i-1]}(t)\right]x^{[i]} \tag{13}$$

where

$$\dot{x}^{[1]} = A(x_0)x^{[1]} + \hat{B}(x_0)x^{[1]}$$

The following theorem gives the convergence condition of this approximation.

Theorem 2: Let κ_1, κ_2, κ_3, κ_4, and κ_5 be finite numbers and $A(x)$ and $B(x)$ satisfy

$$\mu[A(x)] \le \kappa_1, \qquad \forall x \in R^n \tag{A.1}$$

$$\|A(x) - A(y)\| \le \kappa_2\|x - y\|, \qquad \forall x, y \in R^n \tag{A.2}$$

$$\|\hat{B}(x) - \hat{B}(y)\| \le \kappa_3\|x - y\|, \qquad \forall x, y \in R^n \tag{A.3}$$

$$\|\hat{B}(x)\| \le \kappa_4, \qquad \forall x \in R^n \tag{A.4}$$

$$\mu[\hat{B}(x)] \le \kappa_5, \qquad \forall x \in R^n \tag{A.5}$$

and suppose that

$$\gamma_2(t) := \frac{\kappa_1 e^{\kappa_1(t-t_0)}\|x_0\|}{\left\{\kappa_1 + \kappa_4\left(1 - e^{\kappa_1(t-t_0)}\right)\right\}} \times \left\{\frac{1}{2}\kappa_2\kappa_4(t-t_0)^2 + \left(\kappa_2 + \kappa_3 + \frac{\kappa_2\kappa_4}{\kappa_1 - \kappa_5}\right)(t - t_0) + \left(\frac{\kappa_2\kappa_4 + \kappa_3(\kappa_5 - \kappa_1)}{(\kappa_5 - \kappa_1)^2}\right)\left(e^{(\kappa_5-\kappa_1)(t-t_0)} - 1\right)\right\}$$

where $|\gamma_2(t)| < 1$ for $t \in [t_0, T]$. Then the solution of approximating sequence (13) converges to the solution of Eq. (12) on $C([t_0, T], R^n)$.

Proof: The solution to the differential equation of approximating sequence (13) is

$$x^{[i]}(t) = \Phi^{[i-1]}(t, t_0)x_0 + \int_{t_0}^{t} \Phi^{[i-1]}(t, s)\hat{B}\left[x^{[i-1]}(s)\right]x^{[i]}\,\mathrm{d}s$$

Hence,

$$\begin{aligned} x^{[i]}(t) - x^{[i-1]}(t) &= \left[\Phi^{[i-1]}(t, t_0) - \Phi^{[i-2]}(t, t_0)\right]x_0 \\ &+ \int_{t_0}^{t} \Phi^{[i-1]}(t, s)\hat{B}\left[x^{[i-1]}(s)\right]\left[x^{[i]}(s) - x^{[i-1]}(s)\right]\mathrm{d}s \\ &+ \int_{t_0}^{t}\left[\Phi^{[i-1]}(t, s) - \Phi^{[i-2]}(t, s)\right]\hat{B}\left[x^{[i-1]}(s)\right]x^{[i-1]}(s)\,\mathrm{d}s \\ &+ \int_{t_0}^{t} \Phi^{[i-2]}(t, s)\left\{\hat{B}\left[x^{[i-1]}(s)\right] - \hat{B}\left[x^{[i-2]}(s)\right]\right\}x^{[i-1]}(s)\,\mathrm{d}s \end{aligned}$$

From Lemma 1,

$$\begin{aligned} \left\|x^{[i]}(t) - x^{[i-1]}(t)\right\| &\le \kappa_2 e^{\kappa_1(t-t_0)}(t - t_0)\sup_{s\in[t_0,t]}\left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|\|x_0\| + \int_{t_0}^{t} e^{\kappa_1(t-s)}\kappa_4\left\|x^{[i]}(s) - x^{[i-1]}(s)\right\|\mathrm{d}s \\ &+ \int_{t_0}^{t}\kappa_2 e^{\kappa_1(t-s)}(t-s)\sup_{s\in[t_0,t]}\left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|\kappa_4 \\ &\times\left\|x^{[i-1]}(s)\right\|\mathrm{d}s + \int_{t_0}^{t} e^{\kappa_1(t-s)}\kappa_3\left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\| \\ &\times\left\|x^{[i-1]}(s)\right\|\mathrm{d}s \end{aligned}$$

Since

$$\left\|x^{[i]}(t)\right\| \le \left[e^{\kappa_1(t-t_0)} + e^{\kappa_5(t-t_0)}\right]\|x_0\|$$

By virtue of Eq. (8),

$$\begin{aligned} \xi^{[i]}(t) &\le \kappa_2 e^{\kappa_1(t-t_0)}(t-t_0)\|x_0\|\xi^{[i-1]}(t) \\ &- \frac{\kappa_4}{\kappa_1}\left[1 - e^{\kappa_1(t-t_0)}\right]\xi^{[i]}(t) + \frac{1}{2}\kappa_2\kappa_4 e^{\kappa_1(t-t_0)}(t-t_0)^2 \\ &\times\|x_0\|\xi^{[i-1]}(t) + \frac{\kappa_2\kappa_4}{\kappa_1 - \kappa_5}e^{\kappa_1(t-t_0)}(t-t_0)\|x_0\|\xi^{[i-1]}(t) \\ &+ \frac{\kappa_2\kappa_4}{(\kappa_1 - \kappa_5)^2}\left[e^{\kappa_5(t-t_0)} - e^{\kappa_1(t-t_0)}\right]\|x_0\|\xi^{[i-1]}(t) \\ &+ \kappa_3 e^{\kappa_1(t-t_0)}(t-t_0)\|x_0\|\xi^{[i-1]}(t) \\ &+ \frac{\kappa_3}{\kappa_5 - \kappa_1}\left[e^{\kappa_5(t-t_0)} - e^{\kappa_1(t-t_0)}\right]\|x_0\|\xi^{[i-1]}(t) \end{aligned}$$

or

$$\begin{aligned} \left\{1 + \frac{\kappa_4}{\kappa_1}\left[1 - e^{\kappa_1(t-t_0)}\right]\right\}\xi^{[i]}(t) &\le \left\{\frac{1}{2}\kappa_2\kappa_4(t-t_0)^2 + \left(\kappa_2 + \kappa_3 + \frac{\kappa_2\kappa_4}{\kappa_1 - \kappa_5}\right)(t-t_0) + \left(\frac{\kappa_2\kappa_4}{(\kappa_5 - \kappa_1)^2} + \frac{\kappa_3}{\kappa_5 - \kappa_1}\right)\right. \\ &\left.\times\left(e^{(\kappa_5-\kappa_1)(t-t_0)} - 1\right)\right\}e^{\kappa_1(t-t_0)}\|x_0\|\xi^{[i-1]}(t) \end{aligned}$$

Thus,

$$\xi^{[i]}(t) \le \gamma_2(t)\xi^{[i-1]}(t)$$

where

$$\gamma_2(t) := \frac{\kappa_1 e^{\kappa_1(t-t_0)}\|x_0\|}{\left\{\kappa_1 + \kappa_4\left(1 - e^{\kappa_1(t-t_0)}\right)\right\}} \times \left\{\frac{1}{2}\kappa_2\kappa_4(t-t_0)^2 + \left(\kappa_2 + \kappa_3 + \frac{\kappa_2\kappa_4}{\kappa_1 - \kappa_5}\right)(t - t_0) + \left(\frac{\kappa_2\kappa_4 + \kappa_3(\kappa_5 - \kappa_1)}{(\kappa_5 - \kappa_1)^2}\right)\left(e^{(\kappa_5-\kappa_1)(t-t_0)} - 1\right)\right\}$$

and so if $|\gamma_2(t)| < 1$ for $t \in [t_0, T]$, we have $x^{[i]}(t) \to x(t)$ on $C([t_0, T], R^n)$. □

It is shown that the convergence of the approximating sequence is also related to the feedback gain matrix, and it may be possible to satisfy $|\gamma_2(t)| < 1$ by choosing proper control even though the convergence of the approximating sequence for the unforced system is not satisfied, that is, $\gamma_1(t) > 1$.

Instead of applying the control given by Eq. (11), consider now SMC for the nonlinear system. Let the sliding surface be defined by $\sigma(x)$. Then the system motion on the sliding surface is $\sigma(x) = 0$. Suppose that, at time t_s, the state trajectory of the system intercepts the sliding surface and the sliding mode exists for $t \ge t_s$. This implies that $\sigma(x) = 0$, for $\forall t \ge t_s$, and $\dot{\sigma}(x) = 0$, for $\forall t \ge t_s$. Thus, by the solving of the following equation for the control u, the so-called equivalent control is determined:

$$\dot{\sigma}(x) = \frac{\partial\sigma(x)}{\partial x}\dot{x} = \frac{\partial\sigma(x)}{\partial x}[A(x)x + B(x)u] = 0 \tag{14}$$

Suppose that the sliding surface $\sigma(x)$ is chosen such that $[\partial\sigma(x)/\partial x]B(x)$ is nonsingular for all t and x. Then the equivalent control

$$u_{\mathrm{eq}} = -\left(\frac{\partial\sigma(x)}{\partial x}B(x)\right)^{-1}\frac{\partial\sigma(x)}{\partial x}[A(x)x] \tag{15}$$

or

$$u_{\mathrm{eq}} = -K_{\mathrm{eq}}(x)x \tag{16}$$

where the nonlinear state feedback gain matrix is

$$K_{eq} = \left(\frac{\partial\sigma(x)}{\partial x}B(x)\right)^{-1}\frac{\partial\sigma(x)}{\partial x}A(x) \tag{17}$$

On the other hand, the following conditions should be satisfied for the sliding mode:

$$\dot\sigma(x) > 0 \quad \text{if} \quad \sigma(x) < 0$$
$$\dot\sigma(x) < 0 \quad \text{if} \quad \sigma(x) > 0 \tag{18}$$

From Eq. (14),

$$\dot\sigma(x) = \frac{\partial\sigma(x)}{\partial x}A(x) + \frac{\partial\sigma(x)}{\partial x}B(x)u \tag{19}$$

By the choosing of the control as follows:

$$u = -\left(\frac{\partial\sigma(x)}{\partial x}B(x)\right)^{-1}\left\{\frac{\partial\sigma(x)}{\partial x}[A(x)x] + \boldsymbol{k}\,\mathrm{sgn}[\sigma(x)]\right\} \tag{20}$$

Eq. (19) becomes $\dot\sigma(x) = -\boldsymbol{k}\,\mathrm{sgn}[\sigma(x)]$.

By the selecting of each element of the $\boldsymbol{k}$ vector as positive, condition (18) is satisfied. By the use of Eq. (17), control (20) can also be rewritten as

$$u = -K_{eq}(x)x - \left(\frac{\partial\sigma(x)}{\partial x}B(x)\right)^{-1}\boldsymbol{k}\,\mathrm{sgn}[\sigma(x)] \tag{21}$$

Then the nonlinear system (1) becomes

$$\dot x = A(x)x - B(x)K_{eq}x - B(x)\left(\frac{\partial\sigma(x)}{\partial x}B(x)\right)^{-1}\boldsymbol{k}\,\mathrm{sgn}[\sigma(x)]$$

or, in more compact form,

$$\dot x = A_{eq}(x)x + f(x) \tag{22}$$

where

$$A_{eq}(x) = A(x) - B(x)K_{eq}(x)f(x) = -B(x)$$
$$\left(\frac{\partial\sigma(x)}{\partial x}B(x)\right)^{-1}\boldsymbol{k}\,\mathrm{sgn}[\sigma(x)]$$

Now consider the following approximating sequence of linear differential equations for Eq. (22):

$$\dot x^{[1]}(t) = A_{eq}(x_0)x^{[1]}(t) + f(x_0), \qquad x^{[1]}(t_0) = x_0$$
$$\dot x^{[i]}(t) = A_{eq}\left[x^{[i-1]}(t)\right]x^{[i]}(t) + f\left[x^{[i-1]}\right], \qquad x^{[i]}(t_0) = x_0 \tag{23}$$

The solution is

$$x^{[i]}(t) = \Phi^{[i-1]}(t, t_0)x_0 + \int_{t_0}^{t}\Phi^{[i-1]}(t, s)f\left[x^{[i-1]}(s)\right]\mathrm{d}s$$

Hence,

$$x^{[i]}(t) - x^{[i-1]}(t) = \left[\Phi^{[i-1]}(t, t_0) - \Phi^{[i-2]}(t, t_0)\right]x_0$$
$$+ \int_{t_0}^{t}\Phi^{[i-1]}(t, s)\left\{f\left[x^{[i-1]}(s)\right] - f\left[x^{[i-2]}(s)\right]\right\}\mathrm{d}s$$
$$+ \int_{t_0}^{t}\left[\Phi^{[i-1]}(t, s) - \Phi^{[i-2]}(t, s)\right]f\left[x^{[i-2]}(s)\right]\mathrm{d}s \tag{24}$$

The following theorem gives the convergence condition of this approximation.

Theorem 3: Let κ_1, κ_2, κ_3, and κ_4 be finite numbers and $A_{eq}(x)$ and $f(x)$ satisfy

$$\mu[A_{eq}(x)] \le \kappa_1, \qquad \forall x \in R^n \tag{A.1}$$
$$\|A_{eq}(x) - A_{eq}(y)\| \le \kappa_2\|x - y\|, \qquad \forall x, y \in R^n \tag{A.2}$$
$$\|f(x) - f(y)\| \le \kappa_3\|x - y\|, \qquad \forall x, y \in R^n \tag{A.3}$$
$$\|f(x)\| \le \kappa_4, \qquad \forall x \in R^n \tag{A.4}$$

and suppose that

$$\gamma_3(t) := e^{\kappa_1(t-t_0)}(t - t_0)(\kappa_2\|x_0\| + \kappa_2\kappa_4/\kappa_1)$$
$$+ \left(\kappa_2\kappa_4/\kappa_1^2 - \kappa_3/\kappa_1\right)\left[1 - e^{\kappa_1(t-t_0)}\right]$$

where $|\gamma_3(t)| < 1$ for $t \in [t_0, T]$. Then the solution of approximating sequence (23) converges to the solution of (22) on $C([t_0, T], R^n)$.

Proof: From Lemma 1 and Eqs. (8) and (24),

$$\left\|x^{[i]}(t) - x^{[i-1]}(t)\right\| \le \kappa_2 e^{\kappa_1(t-t_0)}(t - t_0)\xi^{[i-1]}(t)\|x_0\|$$
$$- (\kappa_3/\kappa_1)\left[1 - e^{\kappa_1(t-t_0)}\right]\xi^{[i-1]}(t)$$
$$+ (\kappa_2\kappa_4/\kappa_1)e^{\kappa_1(t-t_0)}(t - t_0)\xi^{[i-1]}(t)$$
$$+ \left(\kappa_2\kappa_4/\kappa_1^2\right)\left[1 - e^{\kappa_1(t-t_0)}\right]\xi^{[i-1]}(t)$$

or

$$\xi^{[i]}(t) \le \gamma_3(t)\xi^{[i-1]}(t)$$

where

$$\gamma_3(t) := e^{\kappa_1(t-t_0)}(t - t_0)(\kappa_2\|x_0\| + \kappa_2\kappa_4/\kappa_1)$$
$$+ \left(\kappa_2\kappa_4/\kappa_1^2 - \kappa_3/\kappa_1\right)\left[1 - e^{\kappa_1(t-t_0)}\right]$$

and so if $|\gamma_3(t)| < 1$ for $t \in [t_0, T]$, then $x^{[i]}(t) \to x(t)$ on $C([t_0, T], R^n)$. □

III. SMC for LTV Systems

In the preceding section, the necessary convergence conditions for approximating a class of nonlinear systems as a sequence of LTV differential equations have been investigated. Assuming the convergence conditions that are explored for SMC and for nonlinear state feedback control are satisfied, it can be possible to approximate the nonlinear system and its control as a sequence of LTV models with time-varying control actions. As a result of this approach, the controller can be designed by using the LTV models. These controls are then applied to the nonlinear system until the response of approximations converges to the nonlinear system's response. Because the model now is an LTV system, the SMC design procedure for LTV systems is summarized in this section for the sake of completeness. The LTV system, in general, is given by

$$\dot x = A(t)x + B(t)u \tag{25}$$

where $x \in R^n$, $u \in R^m$, and $A(t)$ and $B(t)$ are time-varying matrices of proper dimensions. It is assumed that $[A(t), B(t)]$ is a controllable pair in the time interval $[t_1, t_2]$. The control function is designed to force the system to an $(n - m)$-dimensional time-varying switching hyperplane of the form

$$\sigma(x, t) = C(t)x \tag{26}$$

and the system is kept on the hyperplane by directing the system states toward it, that is,

$$\dot\sigma(x, t) > 0 \quad \text{if} \quad \sigma(x, t) < 0$$
$$\dot\sigma(x, t) < 0 \quad \text{if} \quad \sigma(x, t) > 0 \tag{27}$$

On the other hand, desired behavior of an LTV system is achieved by deliberate choice of $C(t)$. In most cases, a suitable state transformation puts the problem in an easier form in the selection of

$C(t)$. Thus, we suggest a time-varying nonsingular transformation $z(t) = L(t)x(t)$, where $L(t)$ is a Lyapunov transformation and satisfies

$$L(t)B(t) = \begin{pmatrix} 0 \\ B_2(t) \end{pmatrix}$$

$$L(t)A(t)L^{-1}(t) + \dot{L}(t)L^{-1}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \quad (28)$$

The system in the new coordinates becomes

$$\dot{z}_1 = A_{11}(t)z_1 + A_{12}(t)z_2$$

$$\dot{z}_2 = A_{21}(t)z_1 + A_{22}(t)z_2 + B_2(t)u \quad (29)$$

where $z_1 \in R^{n-m}$, $z_2 \in R^m$, and $B_2(t)$ is an $m \times m$ nonsingular matrix. The time-varying transformation simply divides the system into two subsystems and lumps the control vector into the $z_2(t)$. The switching hyperplane equation (26) can be rewritten in the new coordinates as $\sigma(z, t) = C_1(t)z_1 + C_2(t)z_2$.

Without loss of generality, we assume that $C_2 = I$ and, hence,

$$\sigma(z, t) = C_1(t)z_1 + z_2 \quad (30)$$

The system motion in the sliding mode, that is, motion restricted to the switching surface $\sigma(z, t) = 0$, can be explained by means of the equivalent control method. The existence of a sliding mode implies that $\sigma(z, t) = 0$ and $\dot{\sigma}(z, t) = 0$ for all $t \geq t_s$, where t_s is the time at which the sliding mode begins. Then the equivalent control is

$$u_{\text{eq}} = -B_2^{-1}[\{C_1 A_{11} + A_{21} + \dot{C}_1\}z_1 + \{C_1 A_{12} + A_{22}\}z_2]$$

which yields the so-called equivalent system as

$$\dot{z}_1 = A_{11}(t)z_1 + A_{12}(t)z_2 \quad (31)$$

$$z_2 = -C_1(t)z_1 \quad (32)$$

Hence, the system is described as an $(n - m)$th reduced-order system, where z_2 plays the role of state feedback control. To use the standard linear quadratic regulator problem results, the following lemma is introduced, which is a generalization of Lemma 1.1 of Ref. 1.

Lemma 2: If the system in Eq. (25) is controllable in the time interval $[t_1, t_2]$, then the pair $[A_{11}(t), A_{12}(t)]$ of the reduced-order equivalent system (31) is also controllable in the time interval $[t_1, t_2]$.

Proof: This follows by a simple extension of the proof of Lemma 1.1 of Ref. 1. □

The general control structure that satisfies the stability condition of sliding motion, Eq. (27), can be formulated as

$$u = u_{\text{eq}} + \boldsymbol{k}\,\text{sgn}[\sigma(z, t)] \quad (33)$$

where sgn(·) is the signum function and $\boldsymbol{k}$ is a column vector of dimension m whose elements are $\boldsymbol{k}_i < 0$.

A. Optimal Sliding Surface Design

The dynamic optimization problem for LTV systems is well defined[17] and is to minimize the functional

$$J = \int_0^{t_1} [x^T Q(t)x + u^T P(t)u]\,\mathrm{d}t, \qquad t_1 \leq \infty \quad (34)$$

subject to the system equations (25). Then, the optimal control is

$$u^{\text{op}} = -P^{-1}(t)B^T(t)R(t)x \quad (35)$$

where $R(t)$ is the solution of matrix differential Riccati equation,

$$\dot{R}(t) + R(t)A(t) + A^T(t)R(t) - R(t)B(t)P^{-1}(t)B^T(t)R(t) + Q(t) = 0 \quad (36)$$

with the boundary condition $R(t_1) = 0$. Using the preceding derivation, we shall determine the equation of sliding surface over which the sliding motion is optimal with respect to the criterion

$$J = \int_{t_s}^{t_1} x^T Q(t)x\,\mathrm{d}t, \qquad Q(t) \geq 0 \quad (37)$$

where t_s is the sliding motion starting time. We omit the control term in the functional because the sliding mode motion is control independent and is defined by the equation of discontinuity surfaces. By the applying of the time-varying nonsingular transformation as given by Eq. (28), criterion (37) is written as

$$J = \int_{t_s}^{t_1} \left\{z_1^T Q_{11}(t)z_1 + 2z_1^T Q_{12}(t)z_2 + z_2^T Q_{22}(t)z_2\right\}\mathrm{d}t \quad (38)$$

where

$$[L^{-1}(t)]^T Q(t)L^{-1}(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{pmatrix}$$

By the introduction of a new variable ζ that relates z_1 and z_2

$$\zeta = z_2 + Q_{22}^{-1}(t)Q_{12}^T(t)z_1 \quad (39)$$

the reduced-order system equation (31) and criterion (38) become

$$\dot{z}_1 = \left[A_{11}(t) - A_{12}(t)Q_{22}^{-1}(t)Q_{12}^T(t)\right]z_1 + A_{12}(t)\zeta \quad (40)$$

$$J = \int_{t_s}^{t_1} \left\{z_1^T\left[Q_{11}(t) - Q_{12}(t)Q_{22}^{-1}(t)Q_{12}^T(t)\right]z_1 + \zeta^T Q_{22}(t)\zeta\right\}\mathrm{d}t$$

$$J = \int_{t_s}^{t_1} \left[z_1^T \bar{Q}(t)z_1 + \zeta^T \bar{P}(t)\zeta\right]\mathrm{d}t \quad (41)$$

where $\bar{Q}(t) = Q_{11}(t) - Q_{12}(t)Q_{22}^{-1}(t)Q_{12}^T(t)$ and $\bar{P}(t) = Q_{22}(t)$. By virtue of Eq. (35), the optimum ζ is rewritten as

$$\zeta^{\text{op}} = -\bar{P}^{-1}(t)A_{12}^T(t)R(t)z_1 = -Q_{22}^{-1}(t)A_{12}^T(t)R(t)z_1 \quad (42)$$

where $\dot{R}(t) + R(t)\bar{A}(t) + \bar{A}^T(t)R(t) - R(t)\bar{B}(t)\bar{P}^{-1}(t)\bar{B}^T(t)R(t) + \bar{Q}(t) = 0$, $\bar{A}(t) = A_{11}(t) - A_{12}(t)\,Q_{22}^{-1}(t)Q_{12}^T(t)$, and $\bar{B}(t) = A_{12}(t)$. Using Eqs. (39) and (42), the synthetic control can be written as

$$z_2 = -C_1^{\text{op}}(t)z_1 = -Q_{22}^{-1}(t)\left[A_{12}^T(t)R(t) + Q_{12}^T(t)\right]z_1 \quad (43)$$

which defines the sliding hyperplane as follows:

$$\sigma(z, t) = z_2 + C_1^{\text{op}}(t)z_1 \quad (44)$$

As an application of the theory described in Secs. II and III, a nonlinear missile control problem is studied in the next section.

IV. Missile Dynamics and Control

The nonlinear missile dynamic equations given next have been studied by various authors (for example, see Ref. 15) and represent pitching motion of the missile traveling at Mach 3 at an altitude of 20,000 ft. The same missile dynamics are considered here, and as stated in Ref. 15, they do not represent any particular missile airframe. The aim is to design an autopilot that provides normal acceleration tracking, for the tail-fin-controlled missile. The nonlinear missile dynamics are

$$\dot{\alpha} = f[g\cos(\alpha/f)/WV]F + q, \qquad \dot{q} = fM/I_{yy} \quad (45)$$

where

α = angle of attack, deg
q = pitch rate, deg/s

f = radians-to-degrees conversion, $180/\pi$
g = acceleration of gravity, 32.2 ft/s^2
W = weight, 450 lb
V = speed, 3109.3 ft/s
F = normal force, $C_z QS$, lb
M = pitch moment, $C_m QSD$, ft · lb
I_{yy} = pitch moment of inertia, 182.5 slug · ft^2
Q = dynamic pressure, 6132.8 lb · ft^2
S = reference area, 0.44 ft^2
D = reference diameter, 0.75 ft

The normal force and pitch moment aerodynamic coefficients are approximated for the angle of attack α in the range of ±20 deg as follows:

$$C_z = 0.000103\alpha^3 - 0.00945\alpha|\alpha| - 0.170\alpha - 0.034\delta$$

$$C_m = 0.000215\alpha^3 - 0.0195\alpha|\alpha| + 0.051\alpha - 0.206\delta \quad (46)$$

where δ is fin deflection, in degree.

The missile dynamic equations can be rewritten as follows;

$$\dot{\alpha} = a_{11}(\alpha)\alpha + q + b_1(\alpha)\delta, \qquad \dot{q} = a_{21}(\alpha)\alpha + b_2\delta \quad (47)$$

where

$$a_{11}(\alpha) = \{0.000366354\alpha^2 - 0.03361214|\alpha| - 0.6046628\}\cos(\alpha)$$

$$a_{21}(\alpha) = 0.136606374\alpha^2 - 12.38988|\alpha| + 32.404303$$

$$b_1(\alpha) = -0.1209326\cos(\alpha), \qquad b_2 = -130.887968$$

The autopilot is required to control the normal acceleration of the missile. The normal acceleration is expressed in gravitational acceleration g as $\eta_z = F/W$ or in terms of aerodynamic coefficients as follows:

$$\eta_z = c(\alpha)\alpha + d\delta \quad (48)$$

where

$$c(\alpha) = 0.0006174271\alpha^2 - 0.0566474342|\alpha| - 1.0190543722$$

$$d = -0.203810874$$

In practical applications, the fin deflection is taken as an output of a second-order (or sometimes first-order) filter and combined with the missile dynamics. However, in this study the fin deflection is taken as the control input to the system to reduce the system order and to simplify the simulation model. It is obvious that after determining the necessary control for the missile, that is, fin deflection, one could easily take the fin deflection as a reference output to be followed by another commanded input. In this case the actuator dynamics, whose output is the fin deflection, is controlled by the commanded fin deflection.

On the other hand, we shall use the integral of normal acceleration (hereafter we shall call it normal velocity) as the output of the system so that the derivative of the output, that is, normal acceleration, can be used as an augmented state and the higher derivative of fin deflection can be prevented. In other words, we can still take the fin deflection as the system input. This will also enable us to define the sliding manifold in angle of attack, pitch rate, and normal velocity space. Hence, the output of the system and the tracking error are

$$y = \int \eta_z \, \mathrm{d}t, \qquad e = \int \left(\eta_z - \eta_z^*\right) \mathrm{d}t \quad (49)$$

where η_z^* is desired normal acceleration expressed in gravitational acceleration g. Taking the derivative of Eq. (49), we obtain

$$\dot{e} = \eta_z - \eta_z^* = c(\alpha)\alpha + d\delta - \eta_z^* \quad (50)$$

By combining Eqs. (47) and (50), the following nonlinear third-order system equations are obtained:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} a_{11}(\alpha) & 1 & 0 \\ a_{21}(\alpha) & 0 & 0 \\ c(\alpha) & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ e \end{bmatrix} + \begin{bmatrix} b_1(\alpha) \\ b_2 \\ d \end{bmatrix} \delta + \begin{bmatrix} 0 \\ 0 \\ -\eta_z^* \end{bmatrix} \quad (51)$$

Note that Eq. (51) is in the form of Eq. (1) with an additional term due to the tracking of desired acceleration. The desired normal acceleration is given as step inputs and is changed in every second (see Figs. 1 and 2). Step inputs are smoothed with a time constant of 0.2 s to prevent sudden jumps in the simulations. This is parallel to practical applications because pure step inputs can not be obtained due to time delays. Note that the last term in Eq. (51) is like a bounded disturbance to the system, and the aim here is to satisfy zero tracking error.

The recursive linear approximations of Eq. (51) can be formed, as stated in Sec. II, as follows:

$$\begin{bmatrix} \dot{\alpha}^{[i]} \\ \dot{q}^{[i]} \\ \dot{e}^{[i]} \end{bmatrix} = \begin{bmatrix} a_{11}\left[\alpha^{[i-1]}(t)\right] & 1 & 0 \\ a_{21}\left[\alpha^{[i-1]}(t)\right] & 0 & 0 \\ c\left[\alpha^{[i-1]}(t)\right] & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha^{[i]} \\ q^{[i]} \\ e^{[i]} \end{bmatrix}$$

$$+ \begin{bmatrix} b_1\left[\alpha^{[i-1]}(t)\right] \\ b_2 \\ d \end{bmatrix} \delta^{[i]} + \begin{bmatrix} 0 \\ 0 \\ -\eta_z^* \end{bmatrix} \quad (52)$$

Notice that Eq. (52) is an LTV system and is in the form of Eq. (25). By the application of the Lyapunov transformation as defined by Eq. (28), system (52) becomes

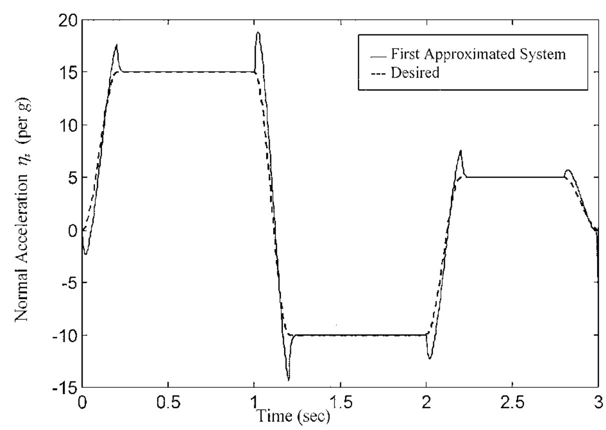

Fig. 1 Normal acceleration η_z of the first approximated system.

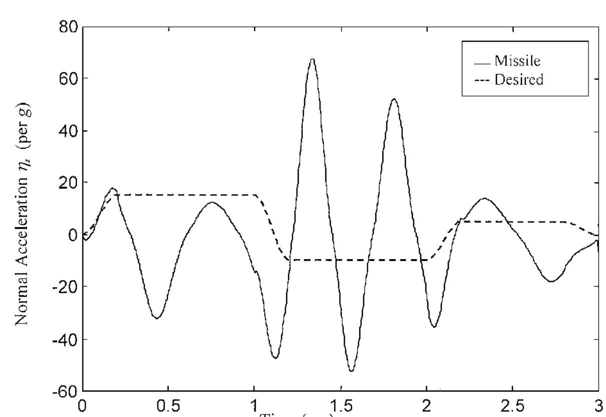

Fig. 2 Normal acceleration η_z response of the missile when the first approximated system model is used.

$$\dot{z}^{[i]} = \begin{bmatrix} \bar{a}_{11}\left[z^{[i-1]}(t)\right] & \cdots & \bar{a}_{13}\left[z^{[i-1]}(t)\right] \\ \vdots & \ddots & \vdots \\ \bar{a}_{31}\left[z^{[i-1]}(t)\right] & \cdots & \bar{a}_{33}\left[z^{[i-1]}(t)\right] \end{bmatrix} z^{[i]} + \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}\delta + \begin{bmatrix} 0 \\ (b_2/d)\eta_z^* \\ -\eta_z^* \end{bmatrix} \tag{53}$$

where the Lyapunov transformation and the corresponding new coordinates are

$$L^{[i]}(t) = \begin{bmatrix} 1 & -\left\{b_1\left[\alpha^{[i-1]}(t)\right]/b_2\right\} & 0 \\ 0 & 1 & -b_2/d \\ 0 & 0 & 1 \end{bmatrix} \tag{54}$$

and

$$z_1^{[i]} = \alpha^{[i]} - \left\{b_1\left[\alpha^{[i-1]}(t)\right]/b_2\right\}q^{[i]}$$

$$z_2^{[i]} = q^{[i]} - (b_2/d)e^{[i]}, \qquad z_3^{[i]} = e^{[i]} \tag{55}$$

The sliding surface is now defined as

$$\sigma^{[i]}\left[z^{[i]}, t\right] = C_1^{[i]}(t)z_1^{[i]} + C_2^{[i]}(t)z_2^{[i]} + z_3^{[i]} \tag{56}$$

On the sliding surface, the system can be expressed by the following reduced-order system equations:

$$\begin{bmatrix} \dot{z}_1^{[i]} \\ \dot{z}_2^{[i]} \end{bmatrix} = \begin{pmatrix} \bar{a}_{11}(t) & \bar{a}_{12}(t) \\ \bar{a}_{21}(t) & \bar{a}_{22}(t) \end{pmatrix} \begin{bmatrix} z_1^{[i]} \\ z_2^{[i]} \end{bmatrix} + \begin{pmatrix} \bar{a}_{13}(t) \\ \bar{a}_{23}(t) \end{pmatrix} z_3^{[i]} + \begin{pmatrix} 0 \\ (b_2/d)\eta_z^* \end{pmatrix} \tag{57}$$

Recall that $z_3^{[i]}$ is like a control input to the reduced-order system and can be interpreted as a state feedback control whose feedback gains are $C_1^{[i]}(t)$ and $C_2^{[i]}(t)$. Thus, the slopes of the sliding hyperplane determine the system behavior. In this study, the slopes of the sliding hyperplane are selected such that the following functional is minimized:

$$J = \int_{t_s}^{t_1} \left([\alpha \quad q \quad e]Q(t) \begin{bmatrix} \alpha \\ q \\ e \end{bmatrix} \right) \mathrm{d}t = \int_{t_s}^{t_1} \left([z_1 \quad z_2]\bar{Q}(t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \bar{P}(t)z_3^2 \right) \mathrm{d}t \tag{58}$$

where $\bar{Q}(t) = \mathrm{diag}\{50, 150 \times b_1^2(t)\}$ and $\bar{P}(t) = 200$.

Note that the weighting matrices are selected just to present the theory, and they do not correspond to any prespecified shape of system response. However, one can construct the performance criteria and select the weighting matrices to shape the desired system output. In Refs. 13 and 17, readers can find some methods of how to choose these weightings.

The optimal sliding surface slopes are then given by

$$C^{\mathrm{op}[i]}(t) = \left[C_1^{\mathrm{op}[i]}(t) \quad C_2^{\mathrm{op}[i]}(t)\right] = \bar{P}^{-1}(t)\bar{B}^T(t)R(t) \tag{59}$$

where

$$\bar{B}(t) = \begin{pmatrix} \bar{a}_{13}(t) \\ \bar{a}_{23}(t) \end{pmatrix}$$

and $R(t)$ is the solution to the following matrix differential Riccati equations:

$$\dot{R}(t) + R(t)\bar{A}(t) + \bar{A}^T(t)R(t) - R(t)\bar{B}(t)\bar{P}^{-1}(t)\bar{B}^T(t)R(t) + \bar{Q}(t) = 0$$

with the boundary condition $R(t_1) = 0$. The system matrix $\bar{A}(t)$ is given by

$$\bar{A}(t) = \begin{pmatrix} \bar{a}_{11}(t) & \bar{a}_{12}(t) \\ \bar{a}_{21}(t) & \bar{a}_{22}(t) \end{pmatrix}$$

The simulation of this control was performed by using MATLAB® software. Because of the term of the signum vector in Eq. (33), the control gives rise to chattering. To prevent chattering and to obtain a continuous control signal, the following smoothed continuous function was used:

$$u = u_{\mathrm{eq}} + \boldsymbol{k}\left(\frac{\sigma(z, t)}{|\sigma(z, t)| + \delta}\right) \tag{60}$$

where each element of $\boldsymbol{k}$ and δ were selected as −1 and 0.01, respectively. The matrix differential Riccati equation was solved by backward integration, and the integration step size for both Riccati equation and simulation was taken as 0.003. The first approximated system was formed by evaluating the nonlinear system at the initial condition of $[\alpha_0 \; q_0 \; e_0] = [0.01 \; -0.5 \; 0]$, which results in an LTI system. The response of the first approximated system and the response of nonlinear missile dynamics to the first approximated control input are given in Figs. 1–4. As noticed, the response of nonlinear system is far away from the approximated system and the missile does not follow the given normal acceleration command (Fig. 2). The first approximated system (LTI system) uses an LTI surface, and this surface does not give desired performance in this particular example. The control input generated from the first approximated system is plotted in Fig. 5. On the other hand, throughout the simulation, it is assumed that there is no control input limitation.

As mentioned earlier, the recursive approximation procedure gives rise to LTV systems after the first approximation. Depending

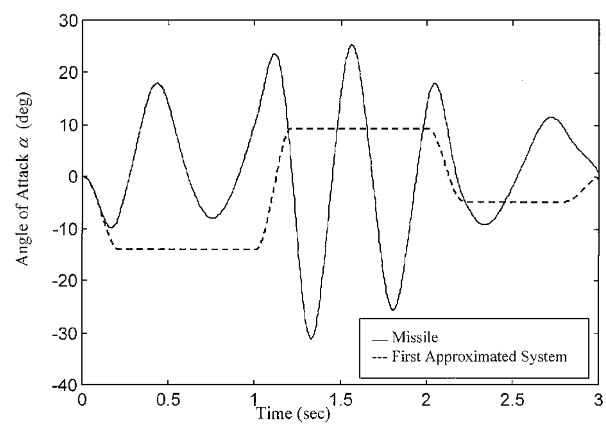

Fig. 3 Angle of attack α of the first approximated system and the missile.

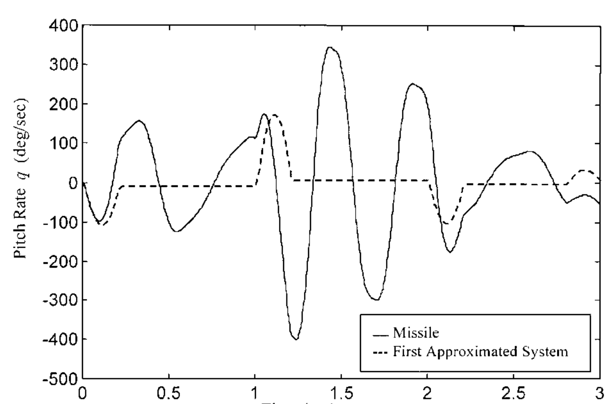

Fig. 4 Pitch rate q of the first approximated system and the missile.

on the convergence rate [Theorem 3, $\gamma_3(t)$], the recursive approximated systems response will converge to the nonlinear system's response. In the simulation studies, five approximations were performed, and satisfactory results were obtained. The fifth approximated system response and the response of nonlinear missile dynamics to the fifth approximated control input are given in Figs. 6–9. It is seen that the response of approximated system is converged to the response of nonlinear missile model, and the control input determined from the fifth approximated LTV system (Fig. 10) makes the nonlinear missile model track the given normal acceleration command. Recall that the weighting matrices in the performance index affect the shape of system response, and thus one might eliminate some of the overshoots in the system acceleration by selecting weighting matrices properly.

The optimal sliding surface slopes for each approximated system are plotted in Figs. 11 and 12. Notice that the slopes of sliding surface also converge to some value (note the bold solid lines in

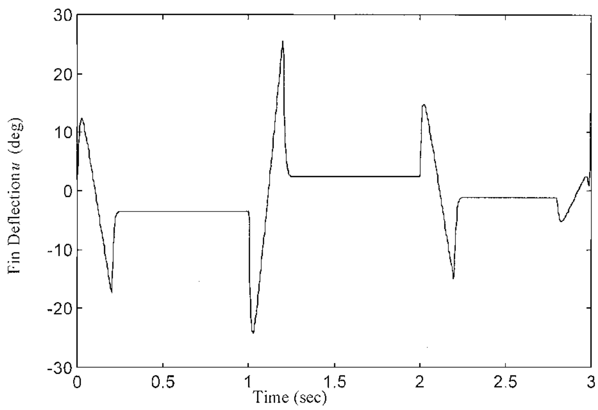

Fig. 5 Control input of the first approximated system (u = fin deflection).

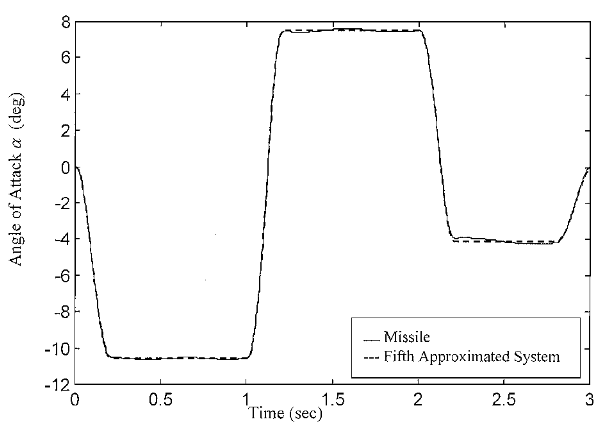

Fig. 6 Angle of attack α of the fifth approximated system and the missile.

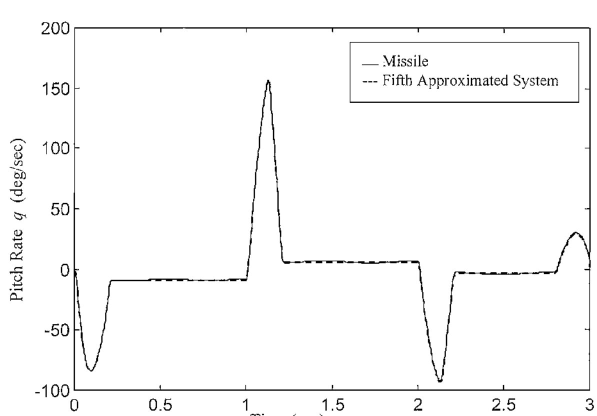

Fig. 7 Pitch rate q of the fifth approximated system and the missile.

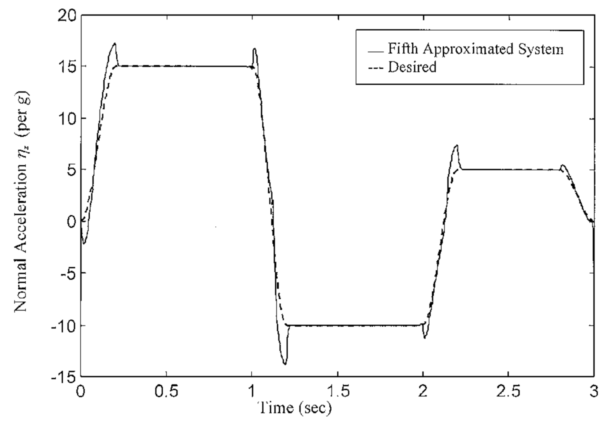

Fig. 8 Normal acceleration η_z of the fifth approximated system.

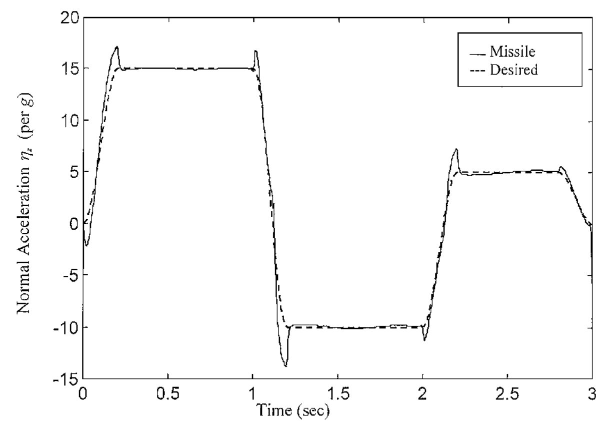

Fig. 9 Normal acceleration η_z response of the missile when the fifth approximated system model is used.

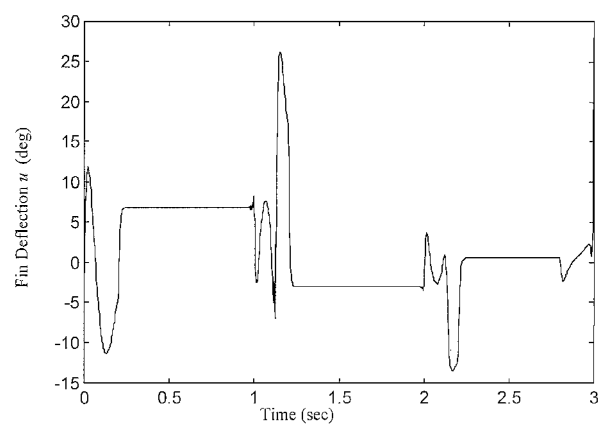

Fig. 10 Control input of the fifth approximated system (u = fin deflection).

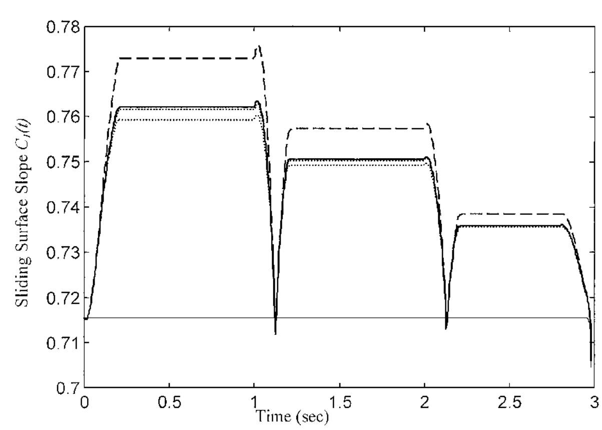

Fig. 11 Time-varying sliding surface slope $C_1(t)$.

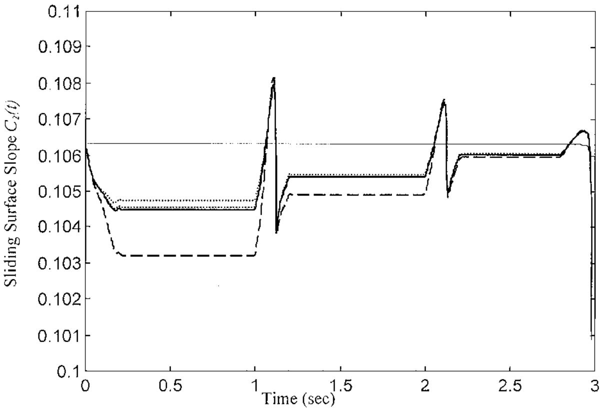

Fig. 12 Time-varying sliding surface slope $C_2(t)$.

Figs. 11 and 12). The sliding surface with optimal slopes for the fifth approximated system can be considered as the optimal sliding surface for the nonlinear system as well. Another point of view is that time-varying sliding surfaces are much more effective then the constant slope sliding surface for this particular nonlinear system.

V. Conclusions

A new approach was presented to design an optimal sliding surface for a class of nonlinear systems. Because the method is based on recursively approximated LTV models of a nonlinear system, the convergence conditions of the approximating sequence were explored first. Then optimal sliding surfaces were designed for each corresponding LTV system. The method was applied to a nonlinear missile dynamics for an autopilot design that provides tracking of a given normal acceleration command. The simulation results show the clear convergence and success of the proposed method. The technique is effective and especially suggested for SMC of a class of nonlinear systems where the design priority is both to select the optimal sliding surface and to minimize a given cost function. However, it is an off-line procedure that requires precomputation and storage of computed data for the control implementation because the nonlinear system is required to be approximated recursively to apply the method. Therefore, further research is essential to develop the technique for control implementation.

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